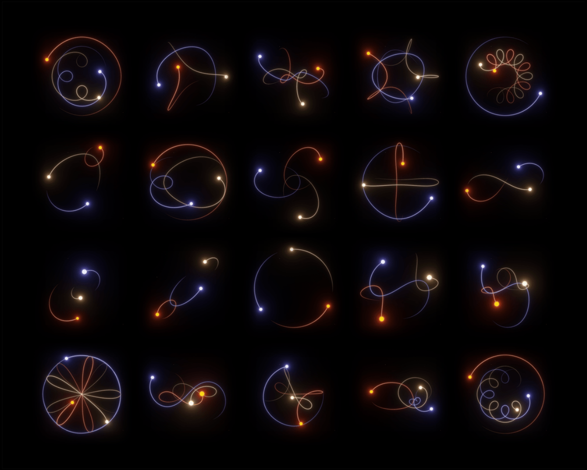
Discovering governing equations from data by sparse identification of nonlinear dynamical systems

Theo Rode April 24, 2024

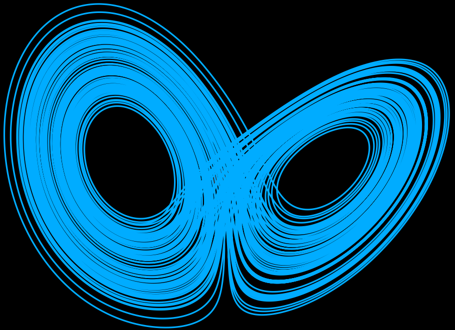
Dynamical Systems



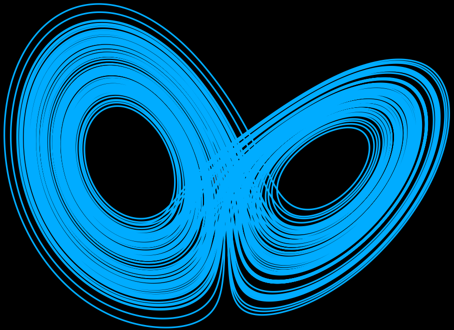
Dynamical System $\rightarrow$ Behvaior

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

$$\longrightarrow$$



Data → Dynamical System



$$\xrightarrow{?} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t))$$

Sparse Identification of Nonlinear Dynamics (SINDy)

(1) Collect data ($\mathbf{X}$) and derivatives ($\dot{\mathbf{X}}$)

(2) Construct function library $\Theta(\mathbf{X})$

(3) Solve sparse regression problem $\dot{\mathbf{X}} = \Theta(\mathbf{X})\boldsymbol{\Xi}$

Data

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^T(t_1) \\ \mathbf{x}^T(t_2) \\ \vdots \\ \mathbf{x}^T(t_m) \end{bmatrix} = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_n(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t_m) & x_2(t_m) & \cdots & x_n(t_m) \end{bmatrix}$$

Derivatives

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{x}}^T(t_1) \\ \dot{\mathbf{x}}^T(t_2) \\ \vdots \\ \dot{\mathbf{x}}^T(t_m) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t_1) & \dot{x}_2(t_1) & \cdots & \dot{x}_n(t_1) \\ \dot{x}_1(t_2) & \dot{x}_2(t_2) & \cdots & \dot{x}_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{x}_1(t_m) & \dot{x}_2(t_m) & \cdots & \dot{x}_n(t_m) \end{bmatrix}$$

Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & & | & | & \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \cdots & \sin\left(\mathbf{X}\right) & \cos\left(\mathbf{X}\right) & \cdots \\ | & | & | & | & & | & | & \end{bmatrix}$$

Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & & | & | & \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \cdots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \cdots \\ | & | & | & | & & | & | & \end{bmatrix}$$

$$\mathbf{X}^{P_2} = \begin{bmatrix} x_1^2(t_1) & x_1(t_1)x_2(t_1) & \cdots & x_1(t_1)x_n(t_1) & x_2^2(t_1) & \cdots & x_n^2(t_1) \\ x_1^2(t_2) & x_1(t_2)x_2(t_2) & \cdots & x_1(t_2)x_n(t_2) & x_2^2(t_2) & \cdots & x_n^2(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^2(t_m) & x_1(t_m)x_2(t_m) & \cdots & x_1(t_m)x_n(t_m) & x_2^2(t_m) & \cdots & x_n^2(t_m) \end{bmatrix}$$

Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & & | & | & \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \cdots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \cdots \\ | & | & | & | & & | & | & \end{bmatrix}$$

$$\sin(\mathbf{X}) = \begin{bmatrix} \sin(x_1(t_1)) & \sin(x_2(t_1)) & \cdots & \sin(x_n(t_1)) \\ \sin(x_1(t_2)) & \sin(x_2(t_2)) & \cdots & \sin(x_n(t_2)) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(x_1(t_m)) & \sin(x_2(t_m)) & \cdots & \sin(x_n(t_m)) \end{bmatrix}$$

Function Library

$$\dot{x}_i(t) = f_i(\mathbf{x}(t)) = a_1 g_1(\mathbf{x}(t)) + a_2 g_2(\mathbf{x}(t)) + \cdots + a_k g_k(\mathbf{x}(t))$$

Function Library

$$\dot{x}_i(t) = f_i(\mathbf{x}(t)) = a_1 g_1(\mathbf{x}(t)) + a_2 g_2(\mathbf{x}(t)) + \cdots + a_k g_k(\mathbf{x}(t))$$

$$\Theta(\mathbf{X}) = \begin{bmatrix} g_1(\mathbf{x}(t_1)) & g_2(\mathbf{x}(t_1)) & \cdots & g_k(\mathbf{x}(t_1)) \\ g_1(\mathbf{x}(t_2)) & g_2(\mathbf{x}(t_2)) & \cdots & g_k(\mathbf{x}(t_2)) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(\mathbf{x}(t_m)) & g_2(\mathbf{x}(t_m)) & \cdots & g_k(\mathbf{x}(t_k)) \end{bmatrix}$$

Sparse Regression

$$\dot{\mathbf{X}} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi}$$

$$\mathbf{\Xi} = \begin{bmatrix} \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \cdots & \boldsymbol{\xi}_n \end{bmatrix}$$

Sparse Regression

```julia
function sparse_regression(Theta, Xdot, lambda, kmax)
    Xi = Theta\Xdot # create initial guess for Xi as just regression
    for _ in 1:kmax
        # find coefficients that are less than lambda
        small_coefficients = abs.(Xi) .< lambda
        Xi[small_coefficients] .= 0 # set small coefficients to 0

        # now individually run regression on non-zero coefficients
        for n in 1:(size(Xdot)[2]) # loop through all state dimensions
            nonsmall = .!small_coefficients[:, n]

            Xi[nonsmall, n] = Theta[:, nonsmall]\Xdot[:, n]
        end # for n
    end # for k
    Xi
end # function sparse_regression
```

Sparse Regression Result

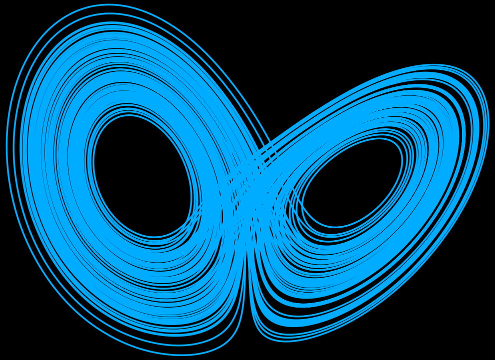
$$\boldsymbol{\Xi} = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_n \end{bmatrix}$$

$$\xi_i = \begin{bmatrix} \xi_{i,1} \\ \xi_{i,2} \\ \vdots \\ \xi_{i,k} \end{bmatrix} \quad \Longrightarrow \quad \dot{x}_i(t) = \xi_{i,1} g_1(\mathbf{x}(t)) + \xi_{i,2} g_2(\mathbf{x}(t)) + \cdots + \xi_{i,k} g_k(\mathbf{x}(t))$$

Most $\xi_{i,j} = 0$.

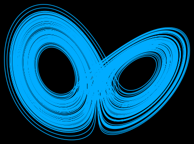
Lorenz System

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$



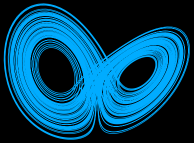
Found Lorenz Systems

Actual



$$\dot{x} = 10(y - x)$$
$$\dot{y} = x(28 - z) - y$$
$$\dot{z} = xy - (8/3)z$$

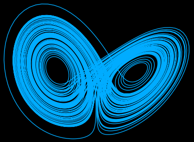
$\eta = 0.01$



$$\dot{x} = 9.976y - 9.947x$$
$$\dot{y} = 27.041x - 0.976xz - 0.669y$$
$$\dot{z} = 0.996xy - 2.650z$$

$$E_{\text{der}} = 0.00136$$

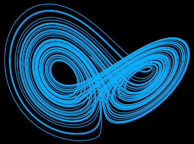
$\eta = 0.1$



$$\dot{x} = 9.838y - 9.806x$$
$$\dot{y} = 25.912x - 0.946xz - 0.433y$$
$$\dot{z} = 0.982xy - 2.613z$$

$$E_{\text{der}} = 0.00349$$

$\eta = 1$



$$\dot{x} = 7.144y - 7.004x$$
$$\dot{y} = 14.982x - 0.635xz + 1.537y$$
$$\dot{z} = 0.839xy - 2.151z - 2.153$$

$$E_{\text{der}} = 0.10119$$

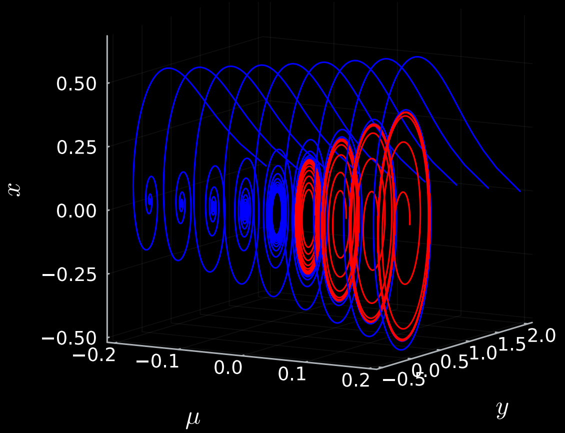
SINDy and Bifurcations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \mu)$$
$$\dot{\mu} = 0$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{\mu_0} \\ \mathbf{X}_{\mu_1} \\ \vdots \\ \mathbf{X}_{\mu_h} \end{bmatrix}$$

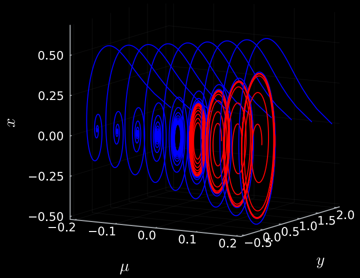
The Hopf Normal Form

$$\dot{x} = \mu x - \omega y - Ax(x^2 + y^2)$$
$$\dot{y} = \omega x + \mu y - Ay(x^2 + y^2)$$



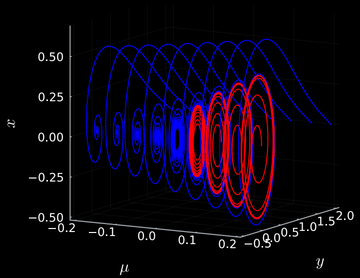
Found Hopf Bifurcation

Actual



$$\dot{x} = \mu x - y - x(x^2 + y^2)$$
$$\dot{y} = x + \mu y - y(x^2 + y^2)$$

$\eta = 0.005$



$$\dot{x} = -0.957y + 0.908\mu x - 0.888x^3 - 1.303xy^2 + 0.769\mu xy^2 - 0.723\mu^2 x^2 y$$
$$\dot{y} = 0.966x + 0.961\mu y - 0.927yx^2 - 0.962y^3$$

$$E_{\text{der}} = 2.682 \times 10^{-6}$$

Concluding Thoughts

- SINDy demonstrates performance in qualitative dynamics recovery
- Concerns over stability with noise and parameters (how do we choose good basis/parameters?)
- Active learning?

Citations

[1] Steven L. Bruton, Joshua L. Proctor, and J. Nathan Kutz. "Discovering governing equations from data by sparse identification of nonlinear dynamical systems". In: *Proceedings of the national academy of sciences* 113.15 (2016), pp. 3932–3937. DOI: `https://doi.org/10.1073/pnas.1517384113`.

[2] Perosello. *20 examples of periodic solutions to the three-body problem.* [Online; accessed April 23, 2024]. 2023. URL: `https://en.wikipedia.org/wiki/Three-body_problem#/media/File:5_4_800_36_downscaled.gif`.