Chapter 1

Introduction into the group theory

1.1 Introduction

In these lectures on group theory and its application to the particle physics there have been considered problems of classification of the particles along representations of the unitary groups, calculations of various characteristics of hadrons, have been studied in some detail quark model. First chapter are dedicated to short introduction into the group theory and theory of group representations. In some datails there are given unitary groups $SU(2)$ and $SU(3)$ which play eminent role in modern particle physics. Indeed $SU(2)$ group is aa group of spin and isospin transforation as well as the base of group of gauge transformations of the electroweak interactons $SU(2) \times U(1)$ of the Salam- Weinberg model. The group $SU(3)$ from the other side is the base of the model of unitary symmetry with three flavours as well as the group of colour that is on it stays the whole edifice of the quantum chromodynamics.

In order to aknowledge a reader on simple examples with the group theory formalism mass formulae for elementary particles would be analyzed in detail. Other important examples would be calculations of magnetic moments and axial-vector weak coupling constants in unitary symmetry and quark models. Formulae for electromagnetic and weak currents are given for both models and problem of neutral currents is given in some detail. Electroweak current of the Glashow-Salam-Weinberg model has been constructed. The notion of colour has been introduced and simple examples with it are given. Introduction of vector bosons as gauge fields are explained. Author would try to write lectures in such a way as to enable an eventual reader to perform calculations of many properties of the elementary particles by oneself.

1.2 Groups and algebras. Basic notions.

Definition of a group

Let be a set of elements G $g_1, g_2, ..., g_n$., with the following properties:

1. There is a multiplication law $g_i g_j = g_l$, and if $g_i, g_j \in G$, then $g_i g_j = g_l \in G$, $i, j, l = 1, 2, ..., n$.

2. There is an associative law $g_i(g_j g_l) = (g_i g_j) g_l$.

3. There exists a unit element e, $eg_i = g_i$, $i = 1, 2, ..., n$.

4. There exists an inverse element g_i^{-1}, $g_i^{-1} g_i = e$, $i = 1, 2, ..., n$.

Then on the set G exists the group of elements $g_1, g_2, ..., g_n$.

As a simple example let us consider rotations on the plane. Let us define a set Φ of rotations on angles ϕ.

Let us check the group properties for the elements of this set.

1. Multiplication law is just a summation of angles: $\phi_1 + \phi_2 = \phi_3 \in \Phi$.

2. Associative law is written as $(\phi_1 + \phi_2) + \phi_3 = \phi_1 + (\phi_2 + \phi_3)$.

3. The unit element is the rotation on the angle $0(+2\pi n)$.

4. The inverse element is the rotation on the angle $-\phi(+2\pi n)$.

Thus rotations on the plane about some axis perpendicular to this plane form the group.

Let us consider rotations of the coordinate axes x, y, z, which define Decartes coordinate system in the 3-dimensional space at the angle θ_3 on the plane xy around the axis z:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta_3 & sin\theta_3 & 0 \\ -sin\theta_3 & cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_3(\theta_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (1.1)$$

Let ϵ be an infinitesimal rotation. Let is expand the rotation matrix $R_3(\epsilon)$

in the Taylor series and take only terms linear in ϵ:

$$R_3(\epsilon) = R_3(0) + \frac{dR_3}{d\epsilon}_{\epsilon=0}\epsilon + O(\epsilon^2) = 1 + iA_3\epsilon + O(\epsilon^2), \tag{1.2}$$

where $R_3(0)$ is a unit matrix and it is introduced the matrix

$$A_3 = -i\frac{dR_3}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.3}$$

which we name 'generator of the rotation around the 3rd axis' (or z-axis). Let us choose $\epsilon = \eta_3/n$ then the rotation on the angle η_3 could be obtained by n-times application of the operator $R_3(\epsilon)$, and in the limit we have

$$R_3(\eta_3) = \lim_{n\to\infty}[R_3(\eta_3/n)]^n = \lim_{n\to\infty}[1 + iA_3\eta_3/n]^n = e^{iA_3\eta_3}. \tag{1.4}$$

Let us consider rotations around the axis y:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} cos\theta_2 & 0 & sin\theta_2 \\ 0 & 1 & 0 \\ -sin\theta_2 & 0 & cos\theta_2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_2(\theta_2)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.5}$$

where a generator of the rotation around the axis y is introduced:

$$A_2 = -i\frac{dR_2}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1.6}$$

Repeat it for the axis x:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta_1 & sin\theta_1 \\ 0 & -sin\theta_1 & cos\theta_1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1(\theta_1)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.7}$$

where a generator of the rotation around the axis xy is introduced:

$$A_1 = -i\frac{dR_1}{d\epsilon}_{\epsilon=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{1.8}$$

Now we can write in the 3-dimensional space rotation of the Descartes coordinate system on the arbitrary angles, for example:

$$R_1(\eta_1)R_2(\eta_2)R_3(\eta_3) = e^{iA_1\eta_1}e^{iA_2\eta_2}e^{iA_3\eta_3}$$

However, usually one defines rotation in the 3-space in some other way, namely by using Euler angles:

$$R(\alpha, \beta, \gamma) = e^{iA_3\alpha}e^{iA_2\beta}e^{iA_3\gamma}$$

functions).

(Usually Cabibbo-Kobayashi-Maskawa matrix is chosen as V_{CKM}=

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}.$$

Here $c_{ij} = cos\theta_{ij}, s_{ij} = sin\theta_{ij}, (i, j = 1, 2, 3)$, while θ_{ij} - generalized Cabibbo angles. How to construct it using Eqs.(1.1, 1.5, 1.7) and setting $\delta_{13} = 0$?)

Generators $A_l, \quad l = 1, 2, 3$, satisfy commutation relations

$$A_i \cdot A_j - A_j \cdot A_i = [A_i, A_j] = i\epsilon_{ijk}A_k, \quad i, j, k = 1, 2, 3,$$

where ϵ_{ijk} is absolutely antisymmetric tensor of the 3rd rang. Note that matrices $A_l, l = 1, 2, 3$, are antisymmetric, while matrices R_l are orthogonal, that is, $R_i^T R_j = \delta_{ij}$, where index T means 'transposition'. Rotations could be completely defined by generators $A_l, l = 1, 2, 3,$. In other words, the group of 3-dimensional rotations (as well as any continuous Lie group up to discrete transformations) could be characterized by its **algebra** , that is by definition of generators $A_l, l = 1, 2, 3,$, its linear combinations and commutation relations.

Definition of algebra

L is the Lie algebra on the field of the real numbers if:

(i) L is a linear space over K (for $x \in$ L the law of multiplication to numbers from the set K is defined), (ii) For $x, y \in$L the commutator is defined as $[x, y]$, and $[x, y]$ has the following properties: $[\alpha x, y] = \alpha[x, y], \quad [x, \alpha y] = \alpha[x, y]$ **at** $\alpha \in K$ **and** $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$**,**

$[x, y_1 + y_2] = [x, y_1] + [x, y_2]$ **for all** $x, y \in$**L;**

$[x, x] = 0$ **for all** $x, y \in$**L;**

$[[x, y]z] + [[y, z]x] + [[z, x]y] = 0$ **(Jacobi identity).**

1.3 Representations of the Lie groups and algebras

Before a discussion of representations we should introduce two notions: isomorphism and homomorphism.

Definition of isomorphism and homomorphism

Let be given two groups, G and G'.

Mapping f of the group G into the group G' is called isomorphism or homomorphis

$$If \quad f(g_1 g_2) = f(g_1) f(g_2) \quad for \quad any \quad g_1 g_2 \in G.$$

This means that if f maps g_1 into g'_1 and g_2 в g'_2, then f also maps $g_1 g_2$ into $g'_1 g'_2$.

However if $f(e)$ maps e into a unit element in G', the inverse in general is not true, namely, e' from G' is mapped by the inverse transformation f^{-1} into $f^{-1}(e')$, named the core (orr nucleus) of the homeomorphism.

If the core of the homeomorphism is e from G such one-to-one homeomorphism is named isomorphism.

Definition of the representation

Let be given the group G and some linear space L. Representation of the group G in L we call mapping T, which to every element g in the group G put in correspondence linear operator $T(g)$ in the space L in such a way that the following conditions are fulfilled:

(1) $T(g_1 g_2) = T(g_1) T(g_2)$ **for all** $g_1, g_2 \in G$,

(2) $T(e) = 1$, **where 1 is a unit operator in** L.

The set of the operators $T(g)$ is homeomorphic to the group G.

Linear space L is called the representation space, and operators $T(g)$ are called representation operators, and they map one-to-one L on L. Because of that the property (1) means that the representation of the group G into L is the homeomorphism of the group G into the G^* (group of all linear operators in в L, with one-to-one correspondence mapping of L in L). If the space L is finite-dimensional its dimension is called dimension of the representation T and named as n_T. In this case choosing in the space L a basis $e_1, e_2, ..., e_n$, current of the Glashow-Salam-Weinberg model has been constructed. The notion of colour has been introduced and simple examples with it are given. Introduction of vector bosons as gauge fields are explained. Author would try to write lectures in such a way as to enable an eventual reader to perform calculations of many properties of the elementary particles by oneself.

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$$t(g) = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix},$$

$$T(g)e_k = \sum t_{ij}(g)e_j, \quad t(e) = 1, \quad t(g_1g_2) = t(g_1)t(g_2).$$

The matrix $t(g)$ is called a representation matrix T. If the group G itself consists from the matrices of the fixed order, then one of the simple representations is obtained at $T(g) = g$ (identical or, better, adjoint representation).

Such adjoint representation has been already considered by us above and is the set of the orthogonal 3×3 matrices of the group of rotations O(3) in the 3-dimensional space. Instead the set of antisymmetrical matrices $A_i, i = 1, 2, 3$ forms adjoint representation of the corresponding Lie agebra. It is obvious that upon constructing all the represetations of the given Lie algebra we indeed construct all the represetations of the corresponding Lie group (up to discrete transformations).

By the transformations of similarity подобия $T'(g) = A^-1T(g)A$ it is possible to obtain from $T(g)$ representation $T'(g) = g$ which is equivalent to it but, say, more suitable (for example, representation matrix can be obtained in almost diagonal form).

Let us define a sum of representations $T(g) = T_1(g) + T_2(g)$ and say that a representation is irreducible if it cannot be written as such a sum (For the Lie group representations is definition is sufficiently correct).

For search and classification of the irreducible representation (IR) Schurr's lemma plays an important role.

Schurr's lemma: Let be given two IR's, $t^\alpha(g)$ and $t^\beta(g)$, of the group G. Any matrix B, such that $Bt^\alpha(g) = t^\beta(g)B$ for all $g \in G$ either is equal to 0 (if $t^\alpha(g)$) and $t^\beta(g)$ are not equivalent) or кратна is proportional to the unit matrix λI.

Therefore if $B \neq \lambda I$ exists which commutes with all matrices of the given representation $T(g)$ it means that this $T(g)$ is reducible. Really,, if $T(g)$ is reducible and has the form

$$T(g) = T_1(g) + T_2(g) = \begin{pmatrix} T_1(g) & 0 \\ 0 & T_2(g) \end{pmatrix},$$

then

$$B = \begin{pmatrix} \lambda_1 I^1 & 0 \\ 0 & \lambda_2 I^2 \end{pmatrix} \neq \lambda I$$

and $[T(g), B] = 0$.

For the group of rotations O(3) it is seen that if $[A_i, B] = 0, i = 1, 2, 3$ then $[R^i, B] = 0$,,i.e., for us it is sufficient to find a matrix B commuting with all the generators of the given representation, while eigenvalues of such matrix operator B can be used for classifications of the irreducible representations (IR's). This is valid for any Lie group and its algebra.

So, we would like to find all the irreducible representations of finite dimension of the group of the 3-dimensional rotations, which can be be reduced to searching of all the sets of hermitian matrices $J_{1,2,3}$ satisfying commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k.$$

There is only one bilinear invariant constructed from generators of the algebra (of the group): $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, for which $[\vec{J}^2, J_i] = 0, i = 1, 2, 3$. So IR's can be characterized by the index j related to the eigenvalue of the operator $\vec{J}^2$.

In order to go further let us return for a moment to the definition of the representation. Operators $T(g)$ act in the linear n-dimensional space L_n and could be realized by $n \times n$ matrices where n is the dimension of the irreducible representation. In this linear space n-dimensional vectors $\vec{v}$ are defined and any vector can be written as a linear combination of n arbitrarily chosen linear independent vectors $\vec{e}_i$, $\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i$. In other words, the space L_n is spanned on the n linear independent vectors $\vec{e}_i$ forming basis in L_n. For example, for the rotation group $O(3)$ any 3-vector can be definned, as we have already seen by the basic vectors

$$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

as $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\vec{x} = xe_x + ye_y + ze_z$. And the 3-dimensional representation (adjoint in this case) is realized by the matrices $R_i, i = 1, 2, 3$. We shall write it now in a different way.

Our problem is to find matrices J_i of a dimension n in the basis of n linear independent vectrs, and we know, first, commutation reltions $[J_i, J_j] = i\epsilon_{ijk}J_k$, and, the second, that IR's can be characterized by $\vec{J}^2$. Besides, it is possible to perform similarity transformation of the Eqs.(1.8,1.6,1.3,) in such a way that one of the matrices, say J_3, becomes diagonal. Then its diagonal elements would be eigenvalues of new basical vectors.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i \\ 0 & 1+i & 0 \\ -i & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1-i & 0 \\ i & 0 & 1 \end{pmatrix} \tag{1.9}$$

$$UA_2U^{-1} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$U(A_1 + A_3)U^{-1} = 2 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$U(A_1 - A_3)U^{-1} = 2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

In the case of the 3-dimensional representation usually one chooses

$$J_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{1.10}$$

$$J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix} \tag{1.11}$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.12}$$

Let us choose basic vectors as

$$|1+1> = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1 \quad 0> = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1-1> = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and

$$J_3|1\ +1>=+|1\ +1>,\quad J_3|1\quad 0>=0|1\ +1>,$$
$$J_3|1\ -1>=-|1\ +1>\ .$$

In the theory of angular momentum these quantities form basis of the representation with the full angular momentum equal to 1. But they could be identified with 3-vector in any space, even hypothetical one. For example, going a little ahead, note that triplet of π-mesons in isotopic space could be placed into these basic vectors:

$$\pi^+,\pi^-,\pi^0\rightarrow|\pi^\pm>=|1\ \pm1>,\quad|\pi^0>=|1\quad 0>\ .$$

Let us also write in some details matrices for $J=2$, i.e., for the representation of the dimension $n=2J+1=5$:

$$J_1=\begin{pmatrix}0&1&0&0&0\\1&0&\sqrt{3/2}&0&0\\0&\sqrt{3/2}&0&\sqrt{3/2}&0\\0&0&\sqrt{3/2}&0&1\\0&0&0&1&0\end{pmatrix}\tag{1.13}$$

$$J_2=\begin{pmatrix}0&-i&0&0&0\\i&0&-i\sqrt{3/2}&0&0\\0&i\sqrt{3/2}&0&-i\sqrt{3/2}&0\\0&0&i\sqrt{3/2}&0&-i\\0&0&0&i&0\end{pmatrix}\tag{1.14}$$

$$J_3=\begin{pmatrix}2&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&0&0&-2\end{pmatrix}\tag{1.15}$$

As it should be these matrices satisfy commutation relations $[J_i,J_j]=i\epsilon_{ijk}J_k, i,j,k=1,2,3$, i.e., they realize representation of the dimension 5 of the Lie algebra corresponding to the rotation group $O(3)$.

Basic vectors can be chosen as:

$$|1+2>=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\quad |1+1>=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\quad |1\quad 0>=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},$$

$$|1-1>=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\quad |1-2>=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}.$$

$J_3|2,+2>=+2|2,+2>$, $J_3|2,+1>=+1|2,+1>$, $J_3|2,0>=0|2,0>$, $J_3|2,-1>=-1|2,-1>$, $J_3|2,-2>=-2|2,-2>$.

Now let us formally put $J=1/2$ although strictly speaking we could not do it. The obtained matrices up to a factor $1/2$ are well known Pauli matrices:

$$J_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$J_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$J_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

These matrices act in a linear space spanned on two basic 2-dimensional vectors

$$\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}.$$

There appears a real possibility to describe states with spin (or isospin) 1/2. But in a correct way it would be possible only in the framework of another group which contains all the representations of the rotation group $O(3)$ plus плюс representations corresponding to states with half-integer spin (or isospin, for mathematical group it is all the same). This group is $SU(2)$.

1.4 Unitary unimodular group SU(2)

Now after learning a little the group of 3-dimensional rotations in which dimension of the minimal nontrivial representation is 3 let us consider more complex group where there is a representation of the dimension 2. For this purpose let us take a set of 2×2 unitary unimodular U,i.e., $U^{\dagger}U = 1$, $detU = 1$. Such matrix U can be written as

$$U = e^{i\sigma_k a_k},$$

$\sigma_k, k = 1, 2, 3$ being hermitian matrices, $\sigma_k^{\dagger} = \sigma_k$, chosen in the form of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $a_k, k = 1, 2, 3$ are arbitrary real numbers. The matrices U form a group with the usual multiplying law for matrices and realize identical (adjoint) represenation of the dimension 2 with two basic 2-dimensional vectors.

Instead Pauli matrices have the same commutation realtions as the generators of the rotation group $O(3)$. Let us try to relate these matrices with a usual 3-dimensional vector $\vec{x} = (x_1, x_2, x_3)$. For this purpose to any vector $\vec{x}$ let us attribute сопоставим a quantity $X = \vec{\sigma}\vec{x}$, :

$$X_b^a = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \quad a, b = 1, 2. \tag{1.16}$$

Its determinant is $detX_b^a = -\vec{x}^2$, that is it defines square of the vector length. Taking the set of unitary unimodular matrices U, $U^{\dagger}U = 1$, $detU = 1$ in 2-dimensional space, let us define

$$X' = U^{\dagger}XU,$$

and $detX' = det(U^{\dagger}XU) = detX = -\vec{x}^2$. We conclude that transformations U leave invariant the vector length and therefore corresponds to rotations in the 3-dimensional space, and note that $\pm U$ correspond to the same rotation. Corresponding algebra $SU(2)$ is given by hermitian matrices $\sigma_k, k = 1, 2, 3$, with the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

where $U = e^{i\sigma_k a_k}$.

And in the same way as in the group of 3-rotations $O(3)$ the representation of lowest dimension 3 is given by three independent basis vectors,for example , x,y,z; in $SU(2)$ 2- dimensional representation is given by two independent basic spinors q^α, $\alpha = 1, 2$ which could be chosen as

$$q^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad q^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The direct product of two spinors q^α and q^β can be expanded into the sum of two irreducible representations (IR's) just by symmetrizing and antisymmetrizing the product:

$$q^\alpha \times q^\beta = \frac{1}{2}\{q^\alpha q^\beta + q^\beta q^\alpha\} + \frac{1}{2}[q^\alpha q^\beta - q^\beta q^\alpha] \equiv T^{\{\alpha\beta\}} + T^{[\alpha\beta]}.$$

Symmetric tensor of the 2nd rank has dimension $d^n_{SS} = n(n+1)/2$ and for $n = 2$ $d^2_{SS} = 3$ which is seen from its matrix representation:

$$T^{\{\alpha\beta\}} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}$$

and we have taken into account that $T_{\{21\}} = T_{\{12\}}$.

Antisymmetric tensor of the 2nd rank has dimension $d^n_{AA} = n(n-1)/2$ and for $n = 2$ $d^2_{AA} = 1$ which is also seen from its matrix representation:

$$T^{[\alpha\beta]} = \begin{pmatrix} 0 & T_{12} \\ -T_{12} & 0 \end{pmatrix}$$

and we have taken into account that $T_{[21]} = -T_{[12]}$ and $T_{[11]} = -T_{[22]} = 0$. Or instead in values of IR dimensions:

$$2 \times 2 = 3 + 1.$$

According to this result absolutely antisymmetric tensor of the 2nd rank $\epsilon_{\alpha\beta}$ ($\epsilon_{12} = -\epsilon_{21} = 1$) transforms also as a singlet of the group $SU(2)$ and we can use it to contract $SU(2)$ indices if needed. This tensor also serves to uprise and lower indices of spinors and tensors in the $SU(2)$.

$$\epsilon_{\alpha'\beta'} = u^{\alpha}_{\alpha'} u^{\beta}_{\beta'} \epsilon_{\alpha\beta} :$$

$$\epsilon_{12} = u^1_1 u^2_2 \epsilon_{12} + u^2_1 u^1_2 \epsilon_{21} = (u^1_1 u^2_2 - u^2_1 u^1_2)\epsilon_{12} = DetU\epsilon_{12} = \epsilon_{12}$$

as $DetU = 1$. (The same for ϵ_{21}.)

1.5 $SU(2)$ as a spinor group

Associating q^{α} with the spin functions of the entities of spin 1/2, $q^1 \equiv |\uparrow\rangle$ and $q^2 \equiv |\downarrow\rangle$ being basis spinors with +1/2 and -1/2 spin projections, correspondingly, (baryons of spin 1/2 and quarks as we shall see later) we can form symmetric tensor $T^{\{\alpha\beta\}}$ with three components

$$T^{\{11\}} = q^1 q^1 \equiv |\uparrow\uparrow\rangle,$$

$$T^{\{12\}} = \frac{1}{\sqrt{2}}(q^1 q^2 + q^2 q^1) \equiv \frac{1}{\sqrt{2}}|(\uparrow\downarrow + \downarrow\uparrow)\rangle,$$

$$T^{\{22\}} = q^2 q^2 \equiv |\downarrow\downarrow\rangle,$$

and we have introduced $1/\sqrt{2}$ to normalize this component to unity.

Similarly for antisymmetric tensor associating again q^{α} with the spin functions of the entities of spin 1/2 let us write the only component of a singlet as

$$T^{[12]} = \frac{1}{\sqrt{2}}(q^1 q^2 - q^2 q^1) \equiv \frac{1}{\sqrt{2}}|(\uparrow\downarrow - \downarrow\uparrow)\rangle, \tag{1.17}$$

and we have introduced $1/\sqrt{2}$ to normalize this component to unity.

Let us for example form the product of the spinor q^{α} and its conjugate spinor q_{β} whose basic vectors could be taken as two rows (1 0) and (0 1). Now expansion into the sum of the IR's could be made by subtraction of a trace (remind that Pauli matrices are traceless)

$$q^{\alpha} \times q_{\beta} = (q^{\alpha} q_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} q^{\gamma} q_{\gamma}) + \frac{1}{2}\delta^{\alpha}_{\beta} q^{\gamma} q_{\gamma} \equiv T^{\alpha}_{\beta} + \delta^{\alpha}_{\beta} I,$$

where T^α_β is a traceless tensor of the dimension $d_V = (n^2-1)$ corresponding to the vector representation of the group $SU(2)$ having at $n = 2$ the dimension 3; I being a unit matrix corresponding to the unit (or scalar) IR. Or instead in values of IR dimensions:

$$2 \times 2 = 3 + 1.$$

The group $SU(2)$ is so little that its representations $T^{\{\alpha\beta\}}$ and T^α_β corresponds to the same IR of dimension 3 while $T^{[\alpha\beta]}$ corresponds to scalar IR together with $\delta^\alpha_\beta I$. For $n \neq 2$ this is not the case as we shall see later.

One more example of expansion of the product of two IR's is given by the product

$$\begin{aligned} T^{[ij]} \times q^k = \\ = \frac{1}{4}(q^iq^jq^k - q^jq^iq^k - q^iq^kq^j + q^jq^kq^i - q^kq^jq^i + q^kq^iq^j) + \\ \frac{1}{4}(q^iq^jq^k - q^jq^iq^k + q^iq^kq^j - q^jq^kq^i + q^kq^jq^i - q^kq^iq^j) = \\ = T^{[ikj]} + T^{[ik]j} \end{aligned} \tag{1.18}$$

or in terms of dimensions:

$$n(n-1)/2 \times n = \frac{n(n^2-3n+2)}{6} + \frac{n(n^2-1)}{3}.$$

For $n = 2$ antisymmetric tensor of the 3rd rank is identically zero. So we are left with the mixed-symmetry tensor $T^{[ik]j}$ of the dimension 2 for $n = 2$, that is, which describes spin 1/2 state. It can be contracted with the the absolutely anisymmetric tensor of the 2nd rank e_{ik} to give

$$e_{ik}T^{[ik]j} \equiv t^j_A,$$

and t^j_A is just the IR corresponding to one of two possible constructions of spin 1/2 state of three 1/2 states (the two of them being antisymmetrized). The state with the $s_z = +1/2$ is just

$$t^1_A = \frac{1}{\sqrt{2}}(q^1q^2 - q^2q^1)q^1 \equiv \frac{1}{\sqrt{2}}|\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle. \tag{1.19}$$

(Here $q^1 =\uparrow$, $q^2 =\downarrow$.)

The last example would be to form a spinor IR from the product of the symmetric tensor $T^{\{ik\}}$ and a spinor q^j.

$$T^{\{ij\}} \times q^k =$$

$$= \frac{1}{4}(q^i q^j q^k + q^j q^i q^k + q^i q^k q^j + q^j q^k q^i + q^k q^j q^i + q^k q^i q^j) + \tag{1.20}$$

$$\frac{1}{4}(q^i q^j q^k + q^j q^i q^k - q^i q^k q^j - q^j q^k q^i - q^k q^j q^i - q^k q^i q^j) =$$

$$= T^{\{ikj\}} + T^{\{ik\}j}$$

or in terms of dimensions:

$$n(n+1)/2 \times n = \frac{n(n^2+3n+2)}{6} + \frac{n(n^2-1)}{3}.$$

Symmetric tensor of the 4th rank with the dimension 4 describes the state of spin S=3/2, (2S+1)=4. Instead tensor of mixed symmetry describes state of spin 1/2 made of three spins 1/2:

$$e_{ij} T^{\{ik\}j} \equiv T^k_S.$$

T^j_S is just the IR corresponding to the 2nd possible construction of spin 1/2 state of three 1/2 states (with two of them being symmetrized). The state with the $s_z = +1/2$ is just

$$T^1_S = \frac{1}{\sqrt{6}}(e_{12} 2 q^1 q^1 q^2 + e_{21}(q^2 q^1 + q^1 q^2) q^1 \equiv \tag{1.21}$$

$$\equiv \frac{1}{\sqrt{6}} |2 \uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle.$$

1.6 Isospin group $SU(2)_I$

Let us consider one of the important applications of the group theory and of its representations in physics of elementary particles. We would discuss classification of the elementary particles with the help of group theory. As a simple example let us consider proton and neutron. It is known for years that proton and neutron have quasi equal masses and similar properties as to strong (or nuclear) interactions. That's why Heisenberg suggested to consider them one state. But for this purpose one should find the group with the (lowest) nontrivial representation of the dimension 2. Let us try (with Heisenberg) to apply here the formalism of the group $SU(2)$ which has as we have seen 2-dimensional spinor as a basis of representation. Let us introduce now a group of isotopic transformations $SU(2)_I$. Now define nucleon as a state with the isotopic spin $I = 1/2$ with two projections (proton with $I_3 = +1/2$ and neutron with $I_3 = -1/2$) in this imagined 'isotopic space' practically in full analogy with introduction of spin in a usual space. Usually basis of the 2-dimensional representation of the group $SU(2)_I$ is written as a isotopic spinor (isospinor)

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$

what means that proton and neutron are defined as

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Representation of the dimension 2 is realized by Pauli matrices 2×2 $\tau_k, k = 1, 2, 3$ (instead of symbols $\sigma_i, i = 1, 2, 3$ which we reserve for spin 1/2 in usual space). Note that isotopic operator $\tau^+ = 1/2(\tau_1 + i\tau_2)$ transforms neutron into proton , while $\tau^- = 1/2(\tau_1 - i\tau_2)$ instead transforms proton into neutron.

It is known also isodoublet of cascade hyperons of spin 1/2 $\Xi^{0,-}$ and masses ~ 1320 MeV. It is also known isodoublet of strange mesons of spin 0 $K^{+,0}$ and masses ~ 490 MeV and antidublet of its antiparticles $\bar{K}^{0,-}$.

And in what way to describe particles with the isospin $I = 1$? Say, triplet of π-mesons π^+, π^-, π^0 of spin zero and negative parity (pseudoscalar mesons) with masses $m(\pi^\pm) = 139,5675 \pm 0,0004$ MeV, $m(\pi^0) = 134,9739 \pm 0,0006$ MeV and practically similar properties as to strong interactions?

In the group of (isotopic) rotations it would be possible to define isotopic vector $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ as a basis (where real pseudoscalar fields $\pi_{1,2}$ are related to charged pions $\pi^{\pm}$ by formula $\pi^{\pm} = \pi^1 \pm i\pi^2$, and $\pi^0 = \pi_3$), generators $A_l, l = 1, 2, 3$, as the algebra representation and matrices $R_l, l = 1, 2, 3$ as the group representation with angles θ_k defined in isotopic space. Upon using results of the previous section we can attribute to isotopic triplet **of the real fields** $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in the group $SU(2)_I$ the basis of the form

$$\pi^a_b = \begin{pmatrix} \pi_3/\sqrt{2} & (\pi_1 - i\pi_2)/\sqrt{2} \\ (\pi_1 + i\pi_2)/\sqrt{2} & -\pi_3/\sqrt{2} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix},$$

where charged pions are described by **complex fields** $\pi^{\pm} = (\pi_1 \mp i\pi_2)/\sqrt{2}$. So, pions can be given in isotopic formalism as 2-dimensional matrices:

$$\pi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi^0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$

which form basis of the representation of the dimension 3 whereas the representation itself is given by the unitary unimodular matrices 2×2 U.

In a similar way it is possible to describe particles of any spin with the isospin $I = 1$. Among meson one should remember isotriplet of the vector (spin 1) mesons $\rho^{\pm,0}$ with masses ~ 760 MeV:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 & \rho^+ \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 \end{pmatrix}. \tag{1.22}$$

Among particles with half-integer spin note, for example, isotriplet of strange hyperons found in early 60's with the spin 1/2 and masses $\sim$1192 MeV $\Sigma^{\pm}$, Σ^0 which can be writen in the $SU(2)$ basis as

$$\begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 & \Sigma^+ \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 \end{pmatrix}, . \tag{1.23}$$

Representation of dimension 3 is given by the same matrices U.

Let us record once more that experimentally isotopic spin I is defined as a number of particles $N = (2I + 1)$ similar in their properties, that is, having the same spin, similar masses (equal at the level of percents) and practically identical along strong interactions. For example, at the mass close to 1115

MeV it was found only one particle of spin 1/2 with strangeness S=-1 - it is hyperon Λ with zero electric charge and mass $1115,63\pm0,05$ MeV. Naturally, isospin zero was ascribed to this particle. In the same way isospin zero was ascribed to pseudoscalar meson $\eta(548)$.

It is known also triplet of baryon resonances with the spin 3/2, strangness S=-1 and masses $M(\Sigma^{*+}(1385)) = 1382,8 \pm 0,4$ MэB, $M(\Sigma^{*0}(1385)) = 1383,7 \pm 1,0$ MэB, $M(\Sigma^{*-}(1385)) = 1387,2 \pm 0,5$ MэB, (resonances are elementary particles decaying due to strong interactions and because of that having very short times of life; one upon a time the question whether they are 'elementary' was discussed intensively) $\Sigma^{*\pm,0}(1385) \to \Lambda^0\pi^{\pm,0}(88 \pm 2\%)$ or $\Sigma^{*\pm,0}(1385) \to \Sigma\pi(12 \pm 2\%)$ (one can find instead symbol $Y_1^*(1385)$ for this resonance).

It is known only one state with isotopic spin $I = 3/2$ (that is on experiment it were found four practically identical states with different charges) : a quartet of nucleon resonances of spin $J = 3/2$ $\Delta^{++}(1232)$, $\Delta^{+}(1232)$, $\Delta^{0}(1232)$, $\Delta^{-}(1232)$, decaying into nucleon and pion (measured mass difference M_{Δ^+}-M_{Δ^0}= 2,7±0,3 MeV). (We can use also another symbol $N^*(1232)$.) There are also heavier 'replics' of this isotopic quartet with higher spins.

In the system $\Xi^{0,-}\pi^{\pm,0}$ it was found only two resonances with spin 3/2 (not measured yet) $\Xi^{*0,-}$ with masses ~ 1520 MeV, so they were put into isodublet with the isospin $I = 1/2$.

Isotopic formalism allows not only to classify practically the whole set of strongly interacting particles (hadrons) in economic way in isotopic multiplet but also to relate various decay and scattering amplitudes for particles inside the same isotopic multiplet.

We shall not discuss these relations in detail as they are part of the relations appearing in the framework of higher symmetries which we start to consider below.

At the end let us remind Gell-Mann–Nishijima relation between the particle charge Q, 3rd component of the isospin I_3 and hypercharge $Y = S + B$, S being strangeness, B being baryon number (+1 for baryons, -1 for antibaryons, 0 for mesons):

$$Q = I_3 + \frac{1}{2}Y.$$

As Q is just the integral over 4th component of electromagnetic current, it means that the electromagnetic current is just a superposition of the 3rd component of isovector current and of the hypercharge current which is isoscalar.

1.7 Unitary symmetry group SU(3)

Let us take now more complex Lie group, namely group of 3-dimensional unitary unimodular matrices which has played and is playing in modern particle physics a magnificient role. This group has already 8 parameters. (An arbitrary complex 3×3 matrix depends on 18 real parameters, unitarity condition cuts them to 9 and unimodularity cuts one more parameter.)

Transition to 8-parameter group $SU(3)$ could be done straightforwardly from 3-parameter group $SU(2)$ upon changing 2-dimensional unitary unimodular matrices U to the 3-dimensional ones and to the corresponding algebra by changing Pauli matrices $\tau_k, k = 1, 2, 3$ to Gell-Mann matrices $\lambda_\alpha, \alpha = 1, .., 8$:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.24}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{1.25}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.26}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.27}$$

$$[\frac{1}{2}\lambda_i, \frac{1}{2}\lambda_j] = i f_{ijk} \frac{1}{2}\lambda_k$$

$f_{123} = 1, f_{147} = \frac{1}{2}, f_{156} = -\frac{1}{2}, f_{246} = \frac{1}{2}, f_{257} = \frac{1}{2}, f_{346} = \frac{1}{2}, f_{367} = -\frac{1}{2}, f_{458} = \frac{\sqrt{3}}{2}, f_{678} = \frac{\sqrt{3}}{2}$.

(In the same way being patient one can construct algebra representation of the dimension n for any unitary group $SU(n)$ of finite n.) These matrices realize 3-dimensional representation of the algebra of the group $SU(3)$ with the basis spinors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Representation of the dimension 8 is given by the matrices 8×8 in the linear space spanned over basis spinors

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$x_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, x_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . x_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, x_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

But similar to the case of $SU(2)$ where any 3-vector can be written as a traceless matrix 2×2, also here any 8-vector in $SU(3)$ $X = (x_1, ..., x_8)$ can be put into the form of the 3×3 matrix:

$$X^\alpha_\beta = \frac{1}{\sqrt{2}} \sum\nolimits_{k=1}^{8} \lambda_k x_k = \tag{1.28}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} x_3 + \frac{1}{\sqrt{3}} x_8 & x_1 - i x_2 & x_4 - i x_5 \\ x_1 + i x_2 & -x_3 + \frac{1}{\sqrt{3}} x_8 & x_6 - i x_7 \\ x_4 + i x_5 & x_6 + i x_7 & -\frac{2}{\sqrt{3}} x_8 \end{pmatrix} .$$

In the left upper angle we see immediately previous expression (1.16) from $SU(2)$.

The direct product of two spinors q^α and q^β can be expanded exactly at the same manner as in the case of $SU(2)$ (but now $\alpha, \beta = 1, 2, 3$) into the sum of two irreducible representations (IR's) just by symmetrizing and antisymmetrizing the product:

$$q^\alpha \times q^\beta = \frac{1}{2}\{q^\alpha q^\beta + q^\beta q^\alpha\} + \frac{1}{2}[q^\alpha q^\beta - q^\beta q^\alpha] \equiv T^{\{\alpha\beta\}} + T^{[\alpha\beta]}. \tag{1.29}$$

Symmetric tensor of the 2nd rank has dimension $d_S^n = n(n+1)/2$ and for $n = 3$ $d_S^3 = 6$ which is seen from its matrix representation:

$$T^{\{\alpha\beta\}} = \begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{12} & T^{22} & T^{23} \\ T^{13} & T^{23} & T^{33} \end{pmatrix}$$

and we have taken into account that $T^{\{ik\}} = T^{\{ki\}} \equiv T^{ik}$ $(i \neq k, i, k = 1, 2, 3)$.

Antisymmetric tensor of the 2nd rank has dimension $d_A^n = n(n-1)/2$ and for $n = 3$ $d_A^3 = 3$ which is also seen from its matrix representation:

$$T^{[\alpha\beta]} = \begin{pmatrix} 0 & t^{12} & t^{13} \\ -t^{12} & 0 & t^{23} \\ -t^{13} & -t^{23} & 0 \end{pmatrix}$$

and we have taken into account that $T^{[ik]} = -T^{[ki]} \equiv t^{ik}$ $(i \neq k, i, k = 1, 2, 3)$ and $T^{[11]} = T^{[22]} = T^{[33]} = 0$.

In terms of dimensions it would be

$$n \times n = n(n+1)/2|_{SS} + n(n-1)/2|_{AA} \tag{1.30}$$

or for $n = 3$ $3 \times 3 = 6 + \bar{3}$.

Let us for example form the product of the spinor q^α and its conjugate spinor q_β whose basic vectors could be taken as three rows $(1 \quad 0 \quad 0)$, $(0 \quad 1 \quad 0)$and $(0 \quad 0 \quad 1)$. Now expansion into the sum of the IR's could be made by subtraction of a trace (remind that Gell-Mann matrices are traceless)

$$q^\alpha \times q_\beta = (q^\alpha q_\beta - \frac{1}{n}\delta^\alpha_\beta q^\gamma q_\gamma) + \frac{1}{n}\delta^\alpha_\beta q^\gamma q_\gamma \equiv T^\alpha_\beta + \delta^\alpha_\beta I,$$

where T^α_β is a traceless tensor of the dimension $d_V = (n^2 - 1)$ corresponding to the vector representation of the group $SU(3)$ having at $n = 3$ the dimension 8; I being a unit matrix corresponding to the unit (or scalar) IR. In terms of dimensions it would be $n \times \bar{n} = (n^2 - 1) + 1_n$ or for $n = 3$ $3 \times \bar{3} = 8 + 1$.

At this point we finish for a moment with a group formalism and make a transition to the problem of classification of particles along the representation of the group $SU(3)$ and to some consequencies of it.